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Alternative Logics: A Book Review*

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A few years ago, Bill Gasarch, editor of the Book Review Column, sent me a copy of the book *Alternative Logics: Do Sciences Need Them?* [28], edited by Paul Weingartner, and asked if I was interested in reviewing it. The book is a collection of essays discussing whether there is a need for logics other than classical logic in various areas of science. Interestingly, I had just finished reading a copy of philosopher Susan Haack's *Deviant Logic* [13], concerning the philosophical foundation of alternative logics. This was fortuitous, because Haack's monograph provided a wonderful introduction to the essays collected in Weingartner's volume.

Let me start by giving a sense of the main question that underlies both books. Generally speaking, when one thinks of logic, one thinks of classical first-order predicate logic. (Throughout, I will take propositional logic as a sublanguage of first-order predicate logic.) Mathematicians sometimes need to go up to second-order, that is, allowing quantification over predicates, in order to express set theory (at least, Zermelo-Fraenkel set theory) and thereby most of modern mathematics. However, at various points in time, philosophers, mathematicians, and scientists have advocated using logics different from classical logic, arguing that the latter was not always appropriate. As a first example, take intuitionism. Intuitionism [16], a philosophical position about the meaning of mathematics, has profound implications on logic as a framework to express mathematics. Without describing intuitionism in depth, one common feature of intuitionism is that it is constructive. As a consequence, the law of excluded middle is often rejected in its full generality: intuitionism does not admit the validity of $A \vee \neg A$; rather, $A \vee \neg A$ is true only when either A can be established explicitly, or $\neg A$ can be established explicitly. Fortunately, much of mathematics survives such a restriction in proving power; but not all—some results are known only via a nonconstructive proof. Thus, the question of which logic is the right logic for mathematics impacts the mathematical results that can be proved.

Around the same time as intuitionism was proposed, another view of classical logic was being questioned. Take propositional logic. There is a well known connection between propositional logic and sets, an instance of Stone Duality [27, 20]. This connection views propositional logic as being the logic of *events*, where an event is a set of states of the world. An elementary proposition A is identified with the set A of states where A is true. A formula $A \wedge B$ is identified with the set of all states where both A and B are true, which is equivalent to $A \cap B$, and similarly $A \vee B$ is identified with the set of all states where either A or B is true, which is equivalent to $A \cup B$.

This correspondence implicitly depends on states behaving as prescribed by Newtonian physics. In particular, Newtonian states satisfy a distributivity property $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ which is inherited by propositional logic. The advent of quantum mechanics has revealed that what we consider states are not so well behaved, and therefore our identification of propositional logic with the logic of events is somewhat misguided at the quantum level. In particular, the distributive property does not always hold because of superposition. Defining logics of events based on quantum states has led to many proposals for quantum logics, starting with Birkhoff and von Neumann [2].

These are but two instances of a general phenomenon. The situation has been compared to the situation in geometry in the late 19th century. After two millennia where geometry was equated with Euclidean geometry, the discovery that the parallel axiom was independent from the other axioms led to the derivation of distinct geometries, as coherent as Euclidean geometry. For a while, the only distinction of Euclidean geometry was that it seemed to be the geometry of the real world. (Until, of course, Einstein blew a hole in this preconception.) Felix Klein's Erlangen programme was a consequence of this new view of geometry, and was an attempt at understanding the plurality of geometry.

The two examples I gave above illustrate that a similar discourse is occurring about the status of logic. And this is the discourse reported in both Haack's and Weingartner's volumes. Before proceeding with the discussion of the volumes, I want to point out that I have borrowed the term *plurality* from Beall and Restall's *Logical Pluralism* [1], which bears on the topic covered in this column. Unfortunately, I have not yet obtained a copy of this book. The teaser is intriguing, however:

> This is our manifesto on *logical pluralism*. We argue that the notion of logical consequence doesn't pin down one deductive consequence relation, but rather, there are many of them. In particular, we argue that broadly classical, intuitionistic and relevant accounts of deductive logic are genuine logical consequence relations. We should not search for *One True Logic*, since there are *Many*.[1]

Susan Haack: *Deviant Logic*

Haack's monograph is a pleasure to read, and provides a reasonably approachable introduction to the topic of alternative logics from a philosophical perspective. At the risk of simplifying her presentation to the point of caricature, Haack is interested in how really different are non-classical logics. From the onset, Haack distinguishes between:

- *Extended logics*, which can be understood as classical logic, extended with features necessary for reasoning about a particular phenomenon not handled directly by classical logic. Modal logics such as temporal logic (operators to reason about time) or epistemic logic (operators to reason about knowledge) are usually taken to be extended logics. Up to differences in syntax and vocabulary, theorems of classical logic remain theorems of an extended logic.

- *Deviant logics*, which can be understood as capturing alternate forms of reasoning. Thus, for instance, Lucasiewicz's three-valued logic or intuitionistic logic embody different logical principles than classical logic. Roughly, some theorems of classical logic cease to be theorems in a deviant logic, again up to differences in syntax and vocabulary.

[1] Taken from the book's web page at `http://consequently.org/writing/logical_pluralism/`.

Note that this classification is not as clear cut as one might hope. There is a lot of wiggle room, for example, in the notion of differences in syntax and vocabulary. (See Felleisen [8] for a computer science perspective on this last topic.)

It is tempting to view extended logics as minor adjustments to our logical apparatus, while viewing deviant logics as representing *rivals* of classical logic as a foundation. However, this picture does not fare so well under close scrutiny, and Haack spends the first half of her book trying to tease out to what extent we should equate deviance (which is a technical concept) with rivalry (which is a psychological attitude with respect to the position of a logic in science), and whether there is any meaning to the notion of rivalry.

The second part of her book illustrates the notion of deviance by examining five canonical examples in detail. Besides intuitionism and quantum mechanics, which I discussed above, Haack also examines future contingents (statements about the future need not be necessarily true or false in the present, so what truth value do we give them?), vagueness (predicates in the real world, for instance color, rarely seem strictly true or false), and singular terms (references need not always denote; what do we do with formulas that refer to the "present king of France"?). The discussion is more often than not illuminating.

It may be useful at this point to ponder to what extent the above classification has anything interesting to say as to how logic is used in computer science. As readers of this column well know, logics in computer science are used in a variety of ways, and my use of the plural here is fully conscious. The article *On the Unusual Effectiveness of Logic in Computer Science* [15] gives an accessible overview of application domains that have especially benefited from a logical approach. Here are some (by no means disjoint or exhaustive) categories we can readily identify:

- Specification and verification: The problem of specifying and formally verifying properties of systems is central to much of computer science. Specification are often expressed in a form of modal logic, and verified by model checking: the logic serves as a formal language for writing down specifications, and verification amounts to checking that a specification A is true in a representation M of the system, that is, $M \models A$ [4, 17]. Other verification approaches rely on interactive theorem proving for extremely expressive logics, usually higher-order—Isabelle [23] is an example of such a framework.

- Artificial intelligence: The intent here is to model various forms of reasoning, and the resulting logics are strongly related to philosophical logics [24]. Additionally, the community studies modal logics for reasoning under uncertainty (for instance, probability, or Dempster-Shafer belief functions) [14], as well as non-classical logics for capturing common sense reasoning, such as default logic [25].

- Descriptive complexity: The complexity of a language (i.e., a set of strings) can be studied by looking at how strings in the language are characterized using formulas; different logics give rise to different languages being expressible [7, 18, 22]. This is essentially a finite form of classical model theory, in which mathematical structures are studied by looking at whether they can be characterized by various fragments of first-order predicate logic. (A recent survey in *SIGACT News* reports on progress in descriptive complexity [19].)

- Computation: there has been many advances in understanding computations using intuitionistic logic. The Curry-Howard isomorphism tells us that proofs in propositional intuitionistic logic can be viewed as programs in a simply-typed lambda calculus that type check at the type

expressed by the proposition [11]. This correspondence can be pushed quite far, as witnessed by many systems such as Coq [6] or Nuprl [5] that take an intuitionistic logic as foundation and can perform *program extraction* to automatically extract from the proof of a property a program satisfying the property.[2]

Of course, just like any formal system, logic can be abused, and computer science has seen a flourishing of logical systems whose foundation can be questioned. Ramsey warns of such abuse in artificial intelligence; after arguing that logical formalisms are necessary to provide a solid foundation for artificial intelligence, he notes:

> At the same time, it looks as though some of the papers using such formalisms are merely disguising the poverty or unoriginality of the work being reported. It seems as though you can make your program respectable if you describe it using a dense logical notation, even if it doesn't actually do anything interesting. [24, p.vii]

Girard is less diplomatic in his somewhat iconoclastic appendix to *Locus Solum* [10]:

> • BROCCOLI LOGICS
> Not as bad as paralogics, Broccoli logics are deductive. The basic idea is to find a logical operation or principle not yet considered... which is not too difficult: call it *Broccoli*. Then the Tarskian machinery works (here the symbol '$\clubsuit$' stands for the syntactical Broccoli):
>
> $$A \clubsuit B \text{ is true if } A \text{ is true } \textbf{Broccoli } B \text{ is true.}$$
>
> If you are smart enough to catch this delicate point, **Broccoli** is the meta of '$\clubsuit$'. Broccoli is equipped with principles that have been never yet considered, typically
>
> $$(A \clubsuit B) \Rightarrow (A \clubsuit (B \clubsuit B))$$
>
> and soundness and completeness are proved with respect to all structures containing a constructor $\heartsuit$ enjoying
>
> $$(a \heartsuit b) \leq (a \heartsuit (b \heartsuit b)).$$
>
> (Hint: to prove completeness, construct the *free Broccolo*.) [10, p.402]

(Girard, by the way, does present an alternative to classical logic in *Locus Solum*, where he introduces a new foundation for reasoning based (very roughly) on a notion of games. This work builds on his previous work on linear logic [9, 11], and provides a third perspective on alternatives to classical logic. Linear logic is a form of substructural logic, characterized by not allowing arbitrary manipulations of formulas in premises of deductions. For instance, linear logic does not allow one to duplicate a formula in the premises of a deduction, so that each use of a formula in a proof must be accounted for exactly. Substructural logics are not discussed in either Haack's book, or in Weingartner's collection. Restall [26] provides a good introduction to substructural logic. Note that the Lambek calculus which is at the basis of categorial grammars in linguistics is a particularly weak form of substructural logic [21, 3].)

Back to Girard's quote. One way to (constructively) read his criticism is that logic should bring something to the table. A logic is frequently prescribed syntactically, by giving formulas and inferences rules and axioms that these formulas should satisfy. The semantics are often an afterthought,

[2]Interestingly, classical logic can also be given a computational interpretation; the law of excluded middle bears a strong relationship with non-local control flow [12].

mathematical structures used to interpret the truth of formulas validating the axioms. The most useful logics, in computer science and elsewhere, tend to have a semantics that is both intuitive (in that one can look at a model and understand the significance of properties of that model) and independently motivated. For instance, in logics for distributed computing and verification, models are often simply derived from program executions, so that models have a meaning independently from their use as structures in which to interpret the logic.

Paul Weingartner: *Alternative Logics*

Weingartner's volume is an edition of essays on the topic of alternative logics, with a particular focus on logics for science. The contributions are revisions of papers presented at the conference "Alternative Logics: Do Sciences Need Them?" of the Académie Internationale de Philosophie des Sciences held at the Institut für Wissenschaftstheory, Internationales Forschungszentrum Salzburg in May 1999. Shapere makes the point clearly in his contribution:

> The question, "Does science need a new logic?" can be interpreted in at least two ways. On the one hand, it can be understood as a question of what Carnap would have called the 'object-level': Do any *specific* areas of science today require a new logic in order to solve *specific* problems arising in those areas? [...] On the other hand, the question can have a 'metalevel' focus, namely, 'Does science (in the *general* sense) require a new logic, or at least a more persistent and competent application of the logic we have, in order to understand its *general* character and procedures?'(p.43).

By and large, contributions in this collection cover both points given by Shapere. The collection has three parts, but the division is not a very crisp one. The first part covers the general concept of alternative logics, essentially at the level of the first part of Haack's monograph, with contributions discussing philosophical implications of alternative logics. The second part focuses on discussions of alternative logics useful for science as a whole, while the third part focuses on discussions of alternative logics prompted by specific applications, such as computer science or quantum mechanics.

Here is an outline of the contributions.

I. **GENERAL TOPICS**

1. **Why is it Logical to Admit Several Logics**
   Agazzi argues for there being many logics, in analogy with Klein's Erlangen's programme in geometry. More precisely, he argues that there is a sense in which there is a single logic, and a sense in which there is a plurality of logics, and that the two sense can co-exist.

2. **Does Metaphysics Need a Non-Classical Logic?**
   Quesada examines the relationship between metaphysics and logic. To illustrate this relationship, note that one role of metaphysics is to explicate what actually exists. This is reflected logically by the extent of the existential and universal quantifiers. In this way, metaphysical claims impact the interpretation of logical constants. Quesada illustrates the relationship through the metaphysics of Plotinus, Hegel, the empiricists (e.g., Mill), and Routly (a proponent of noneism, advocating that the universe contains nonexistent objects).

3. **Logic and the Philosophical Interpretation of Science**
   Shapere revisits the Vienna circle's "logistic" programme of the first half of the twentieth century, which was an attempt to understand science (the whole of science, the enterprise of science) using the then recent formal logic developed by Frege, Whitehead, and Russell, and argues the reasons for the programme's failure.

4. **How Set Theory Impinges on Logic**
   Mosterín examines how logic and set theory are inextricably intertwined. In particular, while set theory (say, ZFC) is in flux, with many unresolved questions, logic is thought to be independent and essentially understood. However, as Mosterín argues, second-order logic (and up to a point, first-order logic) is as open as set theory. In particular, every axiom of ZFC is expressible in second-order logic, and with some work, every axiom of ZFC can be made to correspond to a closed pure second-order logic formula, with the property that the formula is valid if and only if the axiom holds. Thus, the status of validities in second-order logic is intrinsically linked to axioms of ZFC. More generally, what second-order logic *is* depends on what set theory is.

5. **Geometries and Arithmetics**
   Priest re-examines the three a priori sciences of Kant: arithmetic, geometry, and logic. It is by now accepted that geometry is not a priori, in the sense that there is a plurality of geometries, and which one applies to our external world is a contingent fact. Priest argues that arithmetic is similarly contingent, in that it is conceivable to develop different models of arithmetic, alternatives to the standard model—in particular, these models may be inconsistent, but still useful. Interestingly, to reason about such models requires a paraconsistent logic,[3] indicating that not only is arithmetic not a priori, neither is logic.

6. **Remarks on the Criteria of Truth and Models in Science**
   Del Re discusses the role of logic in the sciences of Nature from the side of scientists. Del Re's view is that logic can be taken to be the study of mental operations involved in the attainment of a truth judgment, in particular, procedures by which knowledge is obtained from premises and data, and criteria and rules for deciding the validity of the knowledge so obtained. He further argues that more attention should be given to the role of analogies and true versus reasonable statements.

7. **Significant? Not significant? The Dilemma of Statistical Inference**
   Scardoni very briefly discusses the question of statistical inference and its role in science.

II. **ALTERNATIVE PROPOSALS**

8. **Outline of a Paraconsistent Category Theory**
   Da Costa, Bueno, and Volkov explore the definition of category theory independently from set theory. Starting from classical first-order predicate logic with equality, they derive category theory as a theory in the logic. Then they show how to devise a paraconsistent category theory by taking the underlying logic to be itself paraconsistent. A philosophical discussion of paraconsistency is included, necessary to understand how the theory is to be used.

[3] Very roughly speaking, a logic is paraconsistent if it can be the underlying logic of an inconsistent (admitting both A and $\neg A$ for some A) but nontrivial (not admitting all formulas) theory.

9. **Combinatory Logic, Language, and Cognitive Representation**
   Desclés proposes Church's combinatory logic as a foundation for defining, analyzing, and comparing classical and non-classical logical systems—a prelogic of sorts. The article is in fact a nice presentation of combinatory logic, from a point of view different than the standard computer scientist introduction to combinatory logic.

10. **Extending the Realm of Logic: The Adaptive-Logic Programme**
    Batens proposes a logical approach for capturing actual reasoning, which according to him requires both external dynamics (the possibly of revising premises based on knowledge gained from the external world) as well as internal dynamics (the possibility of revising premises based on knowledge gained from the process of reasoning itself). Abduction, the supposition of a premise based on the explanatory power of that premise, is a typical example of internal dynamics.

11. **Comments on Jaako Hintikka's Post-Tarskian Truth**
    Heinzmann discusses Hintikka's IF-logic (for independence friendly logic). IF-logic, very roughly, allows partially ordered quantifiers, such as:

$$\begin{matrix}\forall x \exists y \\ \forall z \exists u\end{matrix}\ S(x, y, z, u)$$

    which is an attempt at capturing that the y does not depend on the z, and that the u does not depend on the x. Heinzmann addresses Poincaré's foundational remarks about logic as they pertain to IF-logic.

III. ALTERNATIVE LOGICS MOTIVATED BY PROBLEMS OF APPLICATION TO SCIENCE

12. **Applied Logics for Computer Science**
    Gochet and Gribomont describe two applications of first-order logic in computer science. The first is the extension of first-order logic to reason about program first advocated by Hoare, with assertions of the form $\{A\}S\{B\}$, where A and B are formulas and S is a program. The meaning of such assertions is that executing program S in a state satisfying A (the precondition) must either not terminate or terminate in a state satisfying B (the postcondition). These assertions come with inference rules that allow one to derive assertions from assertions about subprograms. The inference rule for looping is especially interesting:

$$\frac{\{I \wedge B\}S\{I\}}{\{I\}\mathsf{while}\ B\ \mathsf{do}\ S\{I \wedge \neg B\}}$$

    which uses an invariant I preserved by every iteration of the loop. The problem of constructing such invariants during a proof of an assertion is examined by Gochet and Gribomont. The second application of first-order logic they describe is logic programming, via a rather nice tutorial illustrating the core ideas of this programming paradigm.

13. **Stochastic *versus* Deterministic Features in Learning Models**
    Stamatescu gives an overview of the debate on the role of randomness and stochastic phenomena in scientific inquiry. Roughly speaking, the two sides of the debate take the view that randomness should either be treated as "statistical noise", or be taken into account directly as a process at work in the model. This debate is illustrated in the context of learning theory.

14. **Praxic Logics**
Finkelstein and Baugh contribute the first article on quantum logic in this collection. They mainly argue for a variant semantics for quantum theory, based on the observation that "pre-quantum physics describes object but quantum physics represents actions" (p. 218). They propose a logic for quantum states corresponding to this variant semantics.

15. **Reasons From Science for Limiting Classical Logic**
Weingartner, after reviewing some existing logical systems for quantum mechanics, proposes an alternative quantum logic that can be viewed as a restriction of first-order predicate logic. Very roughly speaking, inference is restricted so that propositional variables in valid schemas of predicate logic cannot be replaced by arbitrary formulas, but rather must obey a replacement criterion. Similarly, a valid predicate logic formula cannot in general be "reduced" to equivalent simpler formulas (e.g., $C \wedge C$ to C). Weingartner argues that such an approach provides a viable logic system for quantum mechanics.

16. **The Language of Interpretation in Quantum Physics and Its Logic**
It has been widely accepted since the days of the Copenhagen interpretation of quantum mechanics that there is no good language for describing what happens at the quantum level. Omnés argues that there *is* a convenient language for expressing interpretation.

17. **Why Objectivist Programs in Quantum Theory Do Not Need an Alternative Logic**
Cordero critiques the logical turn in foundational quantum theory by questioning some assumption of this programme. This questioning highlights the extent to which proposed quantum logics still embody classical intuitions. He then proceeds to show that these assumptions are mainly dropped from three of the most developed objectivist approaches to quantum theory.

18. **Does Quantum Physics Require a New Logic?**
Mittelstaedt argues that there is no pluralism of logical systems corresponding to different fields of experience (for instance, classical reality versus quantum reality), and that instead there is a hierarchy of logics, with at its base a "true" logic of propositions about physical systems, which he takes to be a quantum logic.

19. **Experimental Approach to Quantum-Logical Connectives**
Stachow devises a process-based semantics for Mittelstaedt's quantum logic (see 18 above), starting from experiments for elementary propositions, and developing experiments yielding logical connectives.

20. **From Semantics to Syntax: Quantum Logic of Observables**
The original presentation of quantum logic by Birkhoff and von Neumann is really an algebraic presentation of a quantum logic, in the same way that Boolean algebras are an algebraic presentation of propositional logic. Vasyukov attempts a syntactical reconstruction of quantum logic corresponding to an algebraic semantics.

21. **An Unsharp Quantum Logic from Quantum Computation**
Cattaneo, Dalla Chiara, and Giuntini take a first step in deriving a quantum logic with a semantics informed by quantum computation. Roughly speaking, the quantum equivalent of

logic gates (operating on qbits, the primitive elements of quantum computation, rather than bits) are taken to provide semantics for the logical connectives. The resulting logic seems to be a weak form of quantum logic.

22. **Quantum Logic and Quantum Probability**
    Beltrametti first reviews algebraic models of events in classical and quantum logic, as well as variations in properties of probability in classical and quantum systems. He then proposes a common extension of classical probability theory and quantum probability theory, by taking convex sets of states as building blocks.

23. **Operator Algebras and Quantum Logic**
    Rédei examines the process of deriving a logic from an algebraic semantics, in particular when the algebra of events is taken to be a general class of non-Boolean lattices arising naturally from certain quantum systems.

As a whole, the papers in the collection tend to be short, and not quite self-contained—the mathematics is often kept short, statements are not proved, and some literature chasing is perforce necessary to understand a paper fully. In particular, had I not gone through Haack's monograph before reading the collection, many of the subtleties would have over my head. There is enormous variation as to the level of technical details present in each contribution, from the more historical and philosophical pieces to the more mathematically-oriented ones. By and large, the more mathematically challenging pieces are those dealing with set theory and with quantum logics. This probably fits correctly with the intended audience, philosophers of science. It is not clear to what extent this collection will speak to an even theoretically-minded computer scientist.

Logic has been called "the calculus of computer science". Weingartner's volume is not calculus for engineers, but calculus for mathematicians. It does not directly impact the daily life of practitioners, but may contribute to a greater understanding of the foundations.

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